A RELATIVISTIC, CAUSAL ACCOUNT OF A SPIN MEASUREMENT

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November 15, 1996

Abstract

An objective account of the action of a Stern-Gerlach apparatus on spin-$\frac{1}{2}$ particles is given, using the Dirac equation. This generalizes earlier work on a causal interpretation of the Pauli equation to the relativistic domain, leading to a more natural choice for the current in the model.

1 Introduction

Dewdney *et al.* [1, 2] have described the action of an inhomogenous magnetic field on an uncharged spin-$\frac{1}{2}$ particle, as in a Stern-Gerlach type spin measurement. Their aim was to demonstrate that there is a causal interpretation of quantum mechanics which is not only consistent with the equations of quantum mechanics, but which quantitatively reproduces the results of measurements made on ensembles of identically prepared systems. According to such an interpretation we may assign well-defined trajectories and spin vectors to particles and study how these interact with the apparatus during the course of a measurement. For a spin measurement, the conventional collapse of the wavefunction onto eigenstates of the spin operators along the field direction is replaced by a continuous evolution of the particle's spin vector towards either alignment or anti-alignment with the field direction. Dewdney *et al.* interpret this alignment as the consequence of 'quantum torques' which act in addition to the usual spin precession in the magnetic field. These quantum torques depend on the wavefunction and hence are sensitive to the whole experimental context.

In this letter we provide a fully relativistic account of a similar spin measurement. It is well known that the current employed by Dewdney *et al.* is inconsistent with that obtained from Dirac theory in the non-relativistic limit, the two differing by a term in the curl of the spin vector [3, 4]. However, the qualitative results of Dewdney *et al.* for the behaviour of the streamlines and the spin vector coincide with our

predictions in the non-relativistic case. Our results demonstrate that the deterministic evolution of the wavefunction suffices to account for the discrete outcomes of a quantum spin measurement. We do not need the relativistic analogue of Bohm/de Broglie theory in order to accept the validity of these results. Besides dealing with a well-defined current, a relativistic treatment is necessary as a foundation for future work extending these ideas to correlated spin measurements made over spacelike intervals, as in EPR-type experiments, so as to take issues of locality properly into account. The treatment presented here is, at the first quantised level, appropriate to scattering from arbitrarily large magnetic field gradients. However we shall see that pair production becomes significant in such cases, demanding ultimately a field-theoretic treatment. With an appropriate change in the electromagnetic interaction term, the present treatment could be extended to model the motion of an electron in an inhomogeneous external magnetic field. This would shed light on Bohr's statement that no measurement of the magnetic moment of the electron could be made, by simply observing the beam splitting on passage through such a field (see *e.g.* appendix to [5]).

In the extension of Bohm's original ideas to include the concept of spin [6], the Pauli spinor is parameterised as

$$|\Phi\rangle = Re^{i\psi/2}\begin{pmatrix} \cos(\theta/2)e^{i\phi/2} \\ i\sin(\theta/2)e^{-i\phi/2} \end{pmatrix} \tag{1.1}$$

where $\{\theta, \phi, \psi\}$ are Euler angles describing the state of rotation of the particle. This spinor is assumed to evolve according to the Pauli equation,

$$i\hbar\partial_t |\Phi\rangle = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 |\Phi\rangle - \mu\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{B}\,|\Phi\rangle\,, \tag{1.2}$$

where μ is the anomalous magnetic moment of the particle. The particle velocity $\boldsymbol{v}$ is defined by $\boldsymbol{J} = \rho\boldsymbol{v}$, where $\rho = \langle\Phi|\Phi\rangle$ and the current $\boldsymbol{J}$ has components J_k given by

$$J_k = \frac{i\hbar}{2m}\left(\langle\partial_k\Phi|\Phi\rangle - \langle\Phi|\partial_k\Phi\rangle\right). \tag{1.3}$$

Note that no spatial integration is implied by the use of bra-ket notation. The spin-vector $\boldsymbol{s}$ has components s_k given by

$$\rho s_k = \frac{\hbar}{2}\langle\Phi|\hat{\sigma}_k|\Phi\rangle = \frac{\hbar}{2}\left(\sin\theta\sin\phi\,, \sin\theta\cos\phi\,, \cos\theta\right) R^2 \tag{1.4}$$

where the $\{\hat{\sigma}_k\}$ are the Pauli spin operators. The Pauli equation can now be used to determine the equation of motion for the spin vector along a chosen streamline. The resulting equation differs from the classical equation by the inclusion of the 'quantum torque' which is a non-local term involving spatial derivatives of the spin current $\rho\boldsymbol{s}$.

$$v_k = \frac{-\hbar}{2m}\left(\partial_k\psi + c_\theta \cdot \partial_k\phi\right)$$

Using these definitions of $\boldsymbol{s}$ and $\boldsymbol{v}$, Dewdney *et al.* calculate the integral curves of the velocity vector (streamlines) and the Euler angles $\{\theta, \phi, \psi\}$ as functions of spatial position and time for a Gaussian wavepacket which is localised along the field direction. They find that the wavefunction evolves in just the way required to give either alignment or anti-alignment with the field direction.

Our treatment of the relativistic problem uses the Dirac equation expressed in the spacetime algebra (STA) formalism of Hestenes [7], which we briefly review in the next section before applying it to the spin measurement in Sections 3 and 4. We believe that the STA is not only a powerful computational tool for discussing quantum mechanics but one which brings genuine new insight into the geometric structure of Dirac theory, implying far-reaching consequences for the quantum theory of measurement. Of course, the results presented here are independent of the representation in which the underlying theory is expressed, but we believe their interpretation is most transparent when considered in the STA. We use natural units ($c = 1, \hbar = 1$) except when presenting the results of our simulations.

2 Dirac Theory in the STA

The spacetime algebra [7, 8] is a real, geometric (Clifford) algebra [9] on Minkowski spacetime. Vectors a, b are equipped with a single product that is associative (and distributive over addition), which we denote ab. It is convenient to introduce four orthonormal basis vectors $\{\gamma_\mu\}$, $\mu = 0, 1, 2, 3$, satisfying

$$\gamma_\mu \cdot \gamma_\nu \equiv \tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = \eta_{\mu\nu} = \operatorname{diag}(+\,-\,-\,-), \tag{2.1}$$

where γ_0 is a timelike unit vector. The algebra of the vectors $\{\gamma_\mu\}$ is the same as that of Dirac's $\hat{\gamma}$-matrices, but they now form an orthonormal basis for spacetime, rather than the four components of a single vector in an internal 'spin-space'.

By repeated multiplication we generate a basis of sixteen elements for the STA,

1	$\{\gamma_\mu\}$	$\{\sigma_k, i\sigma_k\}$	$\{i\gamma_\mu\}$	i
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar,

where $\sigma_k \equiv \gamma_k\gamma_0$ and

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3. \tag{2.2}$$

The grade-4 element i is the highest-grade element in the space, and $i^2 = -1$. However, unlike the uninterpreted scalar imaginary used in conventional quantum mechanics, which we denote by j, it now has a definite geometric interpretation. The timelike bivectors $\{\sigma_k\}$ obey the same algebraic relations as the Pauli spin matrices, but in the STA they represent an orthonormal frame of 'relative' vectors in the Euclidean 3-space relative to an observer whose 4-velocity is γ_0. We shall take this

$\sigma_k\sigma_l = \gamma_k\gamma_0\gamma_l\gamma_0 \overset{?}{=} -\gamma_k\gamma_l\gamma_0\gamma_0 = -\gamma_k\gamma_l =$
$= -(-\delta_{kl} - \gamma_l\gamma_k) = +\delta_{kl} + \gamma_l\gamma_k =$
$= +\delta_{kl} - \sigma_l\sigma_k$

frame as our laboratory frame. To describe observations made in this inertial system we make use of the space-time split: if x is a vector labelling an event in spacetime, this event is seen to occur at time t and position $\boldsymbol{x}$ where

$$x\gamma_0 = t + \boldsymbol{x} = x\cdot\gamma_0 + x\wedge\gamma_0. \tag{2.3}$$

The 'relative' vector $\boldsymbol{x}$ is denoted in bold typeface to distinguish it from (proper) vectors. We denote the derivative with respect to spacetime position by ∇. This operator has the space-time split $\gamma_0\nabla = \partial_t + \boldsymbol{\nabla}$, where $\boldsymbol{\nabla}$ is the usual gradient operator in three-dimensional space, $\boldsymbol{\nabla} \equiv \sigma_k\partial_k$.

Two useful operations are reversion, denoted by a tilde, which reverses the order of 4-vectors in an STA expression, and Hermitian conjugation, denoted by a dagger, which reverses the order of relative 3-vectors in a non-relativistic expression. Angled brackets $\langle\psi\rangle_k$ are used to denote the grade-k part of the enclosed expression, with the subscript zero suppressed when discussing the scalar part. Our notation and conventions follow those in [8, 10].

We shall need the STA representation of the Faraday tensor, in order to discuss the interaction of an anomalous magnetic moment with the electromagnetic field. The components $F_{\mu\nu}$ may be assembled with the set of bivectors $\{\gamma^\mu\wedge\gamma^\nu\}$ to form the Faraday bivector F, given by

$$F \equiv \nabla\wedge A = \tfrac{1}{2}\gamma^\mu\wedge\gamma^\nu F_{\mu\nu}. \tag{2.4}$$

This has the space-time split $F = \boldsymbol{E} + i\boldsymbol{B}$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields in the γ_0 system.

The even sub-algebra of the STA comprises a representation for the Dirac spinor ψ, allowing a reformulation of Dirac theory in the STA. This formalism and some applications to electron physics have been discussed at length in [11], which includes a preliminary summary of the results presented here.

In this representation we may write [12]

$$\psi = (\rho e^{i\beta})^{\frac{1}{2}}R, \tag{2.5}$$

where ρ is a positive scale factor and R is a spacetime rotation (rotor) satisfying $R\tilde{R} = 1$. A direct method of translation between matrix-based approaches and the spacetime algebra of Hestenes is given in Ref. [10]. In the STA, the Dirac equation may be written as [7]

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0. \tag{2.6}$$

The components of the Dirac current assemble with the $\{\gamma_\mu\}$ vectors to give the conserved frame-free current $J = \psi\gamma_0\tilde{\psi}$. The integral curves of J define a set of non-intersecting streamlines in spacetime, whose tangents are always future-pointing

timelike vectors. According to the standard Born interpretation of quantum mechanics, $J^0(\boldsymbol{x},t)$ gives the probability of locating the particle at position $\boldsymbol{x}$ and time t. The conservation of J implies that

$$J^0(\boldsymbol{x}_0,t_0)d^3\boldsymbol{x}_0 = J^0(\boldsymbol{x}_1,t_1)d^3\boldsymbol{x}_1, \tag{2.7}$$

where $(t_0,\boldsymbol{x}_0)$ and $(t_1,\boldsymbol{x}_1)$ are joined by a streamline. The density J^0 thus flows along the flux tubes formed by adjacent streamlines, without 'leaking' out. For this reason, the streamlines are a useful tool for studying the flow of density in dynamic situations, independent of whether one accepts Bohm/de Broglie theory or any other particulate interpretation. It is in this spirit that we plot streamlines for the spin measurement in our simulations. The components $\langle\bar{\psi}|\hat{\gamma}_\mu\hat{\gamma}_5|\psi\rangle$ of the spin current, like those of the Dirac current, assemble with the $\{\gamma_\mu\}$ vectors to give the frame-free spin current $\rho s \equiv \psi\gamma_3\tilde{\psi}$.

The above discussion suggests that the role of the Dirac matrices in the theory is simply to define an orthonormal frame of vectors. It follows that claims of dynamical consequences based solely on their algebraic properties are questionable. One such example is the deduction that the measured velocity of an electron will always yield the speed of light (because the eigenvalues of the 'velocity' operators $\hat{\gamma}_0\hat{\gamma}_k$ are ± 1). To the extent that we may sensibly define a particle velocity v in the Dirac theory, consistency with the laws of quantum mechanics requires that the velocity be in the direction of the Dirac current, *i.e.* $J=\rho v$. If not, particles would end up at positions with probabilities inconsistent with those derived from the Dirac (charge) current. Similar arguments may apply to spin measurements, where the anti-commutation properties of the Dirac matrices (or, non-relativistically, the Pauli matrices) are usually taken to imply that no two components of the spin may be known simultaneously.

Our starting point is a form of the Dirac equation which describes an electrically-neutral spin-$\frac{1}{2}$ particle of anomalous magnetic moment μ. In the conventional representation this equation may be written as [13]

$$(j\hat{\gamma}^\mu\partial_\mu - \tfrac{1}{2}\mu\hat{\sigma}^{\mu\nu}F_{\mu\nu})|\psi(x)\rangle = m|\psi(x)\rangle. \tag{2.8}$$

The operators $\hat{\sigma}^{\mu\nu}$ are defined in terms of the Dirac $\hat{\gamma}$-matrices by

$$\hat{\sigma}^{\mu\nu} \equiv \tfrac{1}{2}j\,[\hat{\gamma}^\mu,\hat{\gamma}^\nu]. \tag{2.9}$$

Translating equation (2.8) into the STA we find the equation

$$\nabla\psi i\sigma_3 - \mu F\psi i\gamma_3 = m\psi\gamma_0. \tag{2.10}$$

Note that there are some inconsistencies in the literature with the sign both of F and the term containing it in the modified Dirac equation. For example, compare Section 9.11 of Greiner [13], with equation (10.81) of Bjorken and Drell [14]. For

this reason, and to make contact with the non-relativistic treatment, we demonstrate that the choice made here is consistent with the Pauli equation in a non-relativistic reduction. Let us define two spinors $\psi_\pm$ where

$$\psi_\pm(\boldsymbol{x},t) \equiv \tfrac{1}{2}(\psi \pm \gamma_0\psi\gamma_0)e^{i\sigma_3 mt}. \tag{2.11}$$

The factor $e^{i\sigma_3 mt}$ removes the contribution of the rest mass to the phase of $\psi_\pm$ and, for sufficiently slow temporal variation in $\psi_\pm$, ensures that we are dealing with positive-energy solutions of (2.10). We identify the spinor ψ_+ with the Pauli spinor Φ, since it is an even element of the geometric algebra of Euclidean 3-space (the Pauli algebra [8]). Equation (2.10) then implies that

$$2m\psi_- \approx -\boldsymbol{\nabla}\Phi i\sigma_3 - \mu \boldsymbol{E}\Phi i\sigma_3, \tag{2.12}$$

where we have neglected the terms $\partial_t\psi_- i\sigma_3$ and $\mu\boldsymbol{B}\psi_-\sigma_3$, which are small compared to $2m\psi_-$ in non-relativistic situations. Using this approximation for ψ_-, we find the equation of motion for the Pauli spinor:

$$\partial_t\Phi i\sigma_3 = -\frac{1}{2m}\Big(\boldsymbol{\nabla}^2\Phi + \mu\boldsymbol{\nabla}(\boldsymbol{E}\Phi) - \mu\boldsymbol{E}\boldsymbol{\nabla}\Phi - \mu^2\boldsymbol{E}^2\Phi\Big) - \mu\boldsymbol{B}\Phi\sigma_3. \tag{2.13}$$

The terms involving $\boldsymbol{E}$ are relativistic corrections to the Pauli equation arising from the motion of the magnetic moment through the electric field. In the limit $c \to \infty$ these terms vanish, leaving the STA form of the Pauli equation. In the STA, the parametrisation of Φ used by Dewdney *et al.* (equation (1.1)) takes the transparent form

$$\Phi = \rho^{1/2}e^{i\sigma_3\phi/2}e^{i\sigma_1\theta/2}e^{i\sigma_3\psi/2}, \tag{2.14}$$

where the rotor term is precisely that needed to describe a rotation in terms of Euler angles.

Using equation (2.12) it is readily shown [3] that a space-time split of the Dirac current gives a 3-current $\boldsymbol{J}$ where

$$m\boldsymbol{J} = -\overset{*}{\boldsymbol{\nabla}}\langle\overset{*}{\Phi}\, i\sigma_3\Phi^\dagger\rangle - \tfrac{1}{2}\boldsymbol{\nabla}\cdot(\Phi i\sigma_3\Phi^\dagger), \tag{2.15}$$

and we have taken $\boldsymbol{E} = 0$. The overstars in this equation denote the action of the gradient operator. The first term is just the STA equivalent of the current employed by Dewdney *et al.* The second term can be written as $\frac{1}{2}\boldsymbol{\nabla}\times(\Phi\Phi^\dagger\boldsymbol{s})$, which includes the curl of the spin 3-current $\Phi\Phi^\dagger\boldsymbol{s}$, obtained as an approximation to the Dirac spin current on splitting in the γ_0 system. This spin 3-current has components given by equation (1.4). Hence the current initially employed by Bohm *et al.* [6] and more recently by Dewdney *et al.* is inconsistent with the non-relativistic reduction of the Dirac theory, although, as already noted, this does not affect their qualitative

conclusions. This fact seems to be widely known [4] but is still ignored in many causal treatments of the Pauli equation. The Dirac current arises naturally as the conserved current conjugate to $U(1)$ gauge transformations of the Dirac field, and hence appears as the source term in the QED Lagrangian. It is thus natural to adopt this definition of the current in any causal approach to Dirac theory.

3 A model for the spin measurement

Suppose we wish to measure the z component of the spin of an uncharged spin-$\frac{1}{2}$ particle. Let the mass of this particle be m and the magnetic moment μ. We observe the deflection of the particle after passage through an inhomogeneous magnetic field directed along our z coordinate axis, and so infer the component of spin along the field direction. We will take our laboratory system to be the γ_0 system.

Let the initial state of the particle be a positive-energy plane-wave solution of the free-particle Dirac equation. We write this state in the form $\psi = \psi_0 e^{-i\sigma_3 p\cdot x}$, where p is the momentum and the constant spinor ψ_0 satisfies

$$p\psi_0 = m\psi_0\gamma_0. \tag{3.1}$$

An explicit form for ψ_0 is (see *e.g.* Ref. [15])

$$\psi_0 = e^{\boldsymbol{u}/2}\Phi, \tag{3.2}$$

where Φ is an arbitrary Pauli spinor and

$$p\gamma_0 = E + \boldsymbol{p} = me^{\boldsymbol{u}}. \tag{3.3}$$

E and $\boldsymbol{p}$ are the energy and momentum of the particle in the γ_0 system. We follow Dewdney *et al.* [1] by assuming that, at time $t = 0$, the particle receives an impulse from a magnetic field gradient, $F = B'zi\sigma_3\delta(t)$, where B' is the field gradient along the z-direction. Clearly this field is not a physically realisable magnetic field, since it has non-vanishing divergence. This could be rectified by the addition of components along the orthogonal directions with suitably chosen spatial gradients. However, we believe that such components would have little effect on the motion along the z direction, and so for simplicity we ignore them. Around this impulse, the equation of motion for the system is given by the split of equation (2.10) in the γ_0 system, which approximates to

$$\partial_t\psi i\sigma_3 = \Delta p z\delta(t)\gamma_3\psi\gamma_3, \tag{3.4}$$

with $\Delta p \equiv \mu B'$. It follows that

$$\partial_t(\psi \pm \gamma_3\psi\gamma_3) = \mp\Delta p z\delta(t)(\psi \pm \gamma_3\psi\gamma_3)i\sigma_3. \tag{3.5}$$

We may summarise the effect of the impulse by

$$(\psi \pm \gamma_3 \psi \gamma_3)_{t=0_+} = (\psi \pm \gamma_3 \psi \gamma_3)_{t=0_-} e^{\mp \Delta p z i \sigma_3}, \tag{3.6}$$

where $t = 0_-$ is just before the impulse and $t = 0_+$ just after. If we decompose the spinor ψ_0 into

$$\psi^\uparrow \equiv \tfrac{1}{2}(\psi_0 - \gamma_3 \psi_0 \gamma_3), \quad \psi^\downarrow \equiv \tfrac{1}{2}(\psi_0 + \gamma_3 \psi_0 \gamma_3) \tag{3.7}$$

we see that, immediately after the shock, ψ is given by

$$\psi = \psi^\uparrow e^{i\sigma_3(\boldsymbol{p}\cdot\boldsymbol{x} + \Delta p z)} + \psi^\downarrow e^{i\sigma_3(\boldsymbol{p}\cdot\boldsymbol{x} - \Delta p z)}. \tag{3.8}$$

The spatial dependence of ψ is now appropriate to two different values of the 3-momentum $\boldsymbol{p}^\uparrow$ and $\boldsymbol{p}^\downarrow$, where

$$\boldsymbol{p}^\uparrow \equiv \boldsymbol{p} + \Delta p \sigma_3, \quad \boldsymbol{p}^\downarrow \equiv \boldsymbol{p} - \Delta p \sigma_3. \tag{3.9}$$

For $t > 0$ we have $F = 0$, so that ψ propagates as a linear combination of four plane-wave states

$$\psi = \psi_+^\uparrow e^{-i\sigma_3 p^\uparrow \cdot x} + \psi_-^\uparrow e^{i\sigma_3 \bar{p}^\uparrow \cdot x} + \psi_+^\downarrow e^{-i\sigma_3 p^\downarrow \cdot x} + \psi_-^\downarrow e^{i\sigma_3 \bar{p}^\downarrow \cdot x}, \tag{3.10}$$

where

$$p^\uparrow \gamma_0 = E^\uparrow + \boldsymbol{p}^\uparrow \tag{3.11}$$

$$\bar{p}^\uparrow \gamma_0 = E^\uparrow - \boldsymbol{p}^\uparrow, \tag{3.12}$$

and

$$E^\uparrow = (m^2 + \boldsymbol{p}^{\uparrow 2})^{1/2}, \quad E^\downarrow = (m^2 + \boldsymbol{p}^{\downarrow 2})^{1/2}, \tag{3.13}$$

with similar expressions for $p^\downarrow$. Each term in equation (3.10) must separately satisfy the free-particle Dirac equation, so that

$$p^\uparrow \psi_+^\uparrow = m \psi_+^\uparrow \gamma_0 \tag{3.14}$$

$$-\bar{p}^\uparrow \psi_-^\uparrow = m \psi_-^\uparrow \gamma_0, \tag{3.15}$$

which are satisfied together with the condition

$$\psi^\uparrow = \psi_+^\uparrow + \psi_-^\uparrow. \tag{3.16}$$

These equations hold also for $\psi^\downarrow$, $\psi_+^\downarrow$ and $\psi_-^\downarrow$. From these conditions we find that

$$\psi_+^\uparrow = \frac{1}{2E^\uparrow}(p^\uparrow \gamma_0 \psi^\uparrow + m \gamma_0 \psi^\uparrow \gamma_0), \tag{3.17}$$

$$\psi_-^\uparrow = \frac{1}{2E^\uparrow}(\bar{p}^\uparrow \gamma_0 \psi^\uparrow - m \gamma_0 \psi^\uparrow \gamma_0), \tag{3.18}$$

with the expressions for $\psi_{\pm}^{\downarrow}$ following similarly.

We see that the effect of the impulse on a monochromatic plane wave is to split it into four plane waves, each with distinct phase velocity. The first two terms in equation (3.10) have phase velocities in the directions $\pm(\boldsymbol{p} + \Delta p\sigma_3)$ respectively, whilst the last two are in the directions $\pm(\boldsymbol{p} - \Delta p\sigma_3)$. A particle incident on the apparatus could be modelled as a wavepacket, formed by superposing the incident plane-wave states above. This packet splits into four packets as a result of the $\boldsymbol{B}$-field gradient. The two transmitted packets, which are composed of positive-energy plane-waves, move apart along the field direction until eventually they no longer overlap significantly, at which point detection of the particle at a screen would give an unambiguous result for the measurement. The two reflected wavepackets are composed of negative-energy plane-waves. The appearance of antiparticle states must ultimately be attributed to pair production in the magnetic field, and is significant only in large magnetic field gradients. A rigorous treatment of this phenomenon requires a multiparticle approach, which we shall not attempt here.

We now verify that in the non-relativistic limit, this causal approach yields the same probabilities for a 'spin-up' and 'spin-down' result as conventional operator methods. In this limit, in which we neglect pair production, the negative energy terms in equation (3.10) are negligible and so we take $\psi_{+}^{\uparrow} = \psi^{\uparrow}$ and $\psi_{+}^{\downarrow} = \psi^{\downarrow}$. If the initial state is a wavepacket composed of positive-energy plane-wave states, each having a weight $g(\boldsymbol{p})$, then after passage through the magnetic field we have

$$\psi(\boldsymbol{x},t) \equiv \psi_u + \psi_d = \int d^3\boldsymbol{p}\, g(\boldsymbol{p}) \left(\psi^{\uparrow} e^{-i\sigma_3 p^{\uparrow}\cdot x} + \psi^{\downarrow} e^{i\sigma_3 p^{\downarrow}\cdot x}\right), \tag{3.19}$$

where ψ_u denotes the packet moving up the field gradient and ψ_d the downwards moving packet. For sufficiently large t the overlap of these packets is small and may be neglected in calculating the density J^0. In this approximation J^0 is given by

$$J^0 = \psi_u \gamma_0 \tilde{\psi}_u + \psi_d \gamma_0 \tilde{\psi}_d. \tag{3.20}$$

The probability P_u of obtaining the 'spin-up' result is simply equal to the probability of finding the particle in the region $z > 0$, which is given by integrating J^0 over this region. The contribution from ψ_d may be neglected since this packet is localised in the region $z < 0$. On performing the spatial integration we find that

$$P_u \propto \int d^3\boldsymbol{p}\, g(\boldsymbol{p})^2 (\psi^{\uparrow}\gamma_0\tilde{\psi}^{\uparrow})\cdot\gamma_0, \tag{3.21}$$

with a similar expression for the probability P_d of the 'spin-down' outcome. Using equations (3.2) and (3.7), it is straightforward to show that

$$(\psi^{\uparrow}\gamma_0\tilde{\psi}^{\uparrow})\cdot\gamma_0 = 2\Phi^{\dagger}\Phi(E(\boldsymbol{p})/m + \boldsymbol{s}\cdot\sigma_3), \tag{3.22}$$

$$(\psi^{\downarrow}\gamma_0\tilde{\psi}^{\downarrow})\cdot\gamma_0 = 2\Phi^{\dagger}\Phi(E(\boldsymbol{p})/m - \boldsymbol{s}\cdot\sigma_3), \tag{3.23}$$

where $\Phi^\dagger \Phi \boldsymbol{s} = \Phi \sigma_3 \Phi^\dagger$. If we take the Pauli spinor Φ to be of the form

$$\Phi = \alpha - \beta i \sigma_2, \tag{3.24}$$

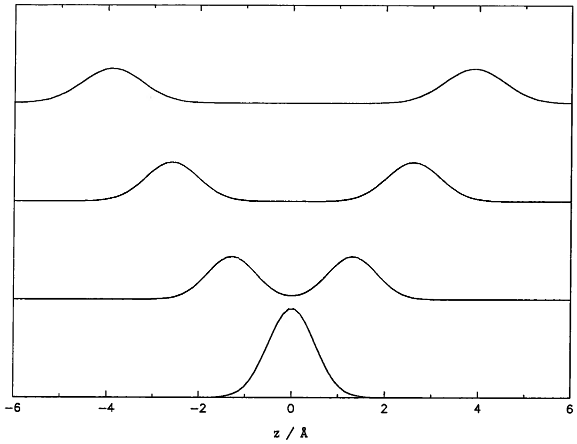
and set $E \approx m$ for all momentum components of the non-relativistic wavepackets, we recover the expected result that the ratio P_u/P_d of the probabilities of the two outcomes is equal to $|\alpha|^2/|\beta|^2$.

4 Wavepacket Simulations

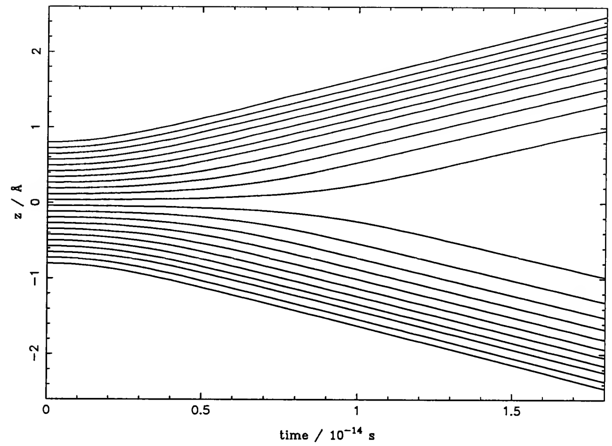
For computational simplicity we take the incident particle to be localised along the field direction only, and with no momentum components transverse to the field. This two-dimensional (one 3-space and the time coordinate) setup was used by Dewdney *et al.* [1] and is sufficient to demonstrate the essential features of the measurement process. The main deficiency of this model is that all four packets have group velocities along the field direction.

We take ψ_0 to be of the form (3.2), with $\boldsymbol{u} = u\sigma_3$ and Φ given by equation (3.24). The parameters α and β are real scalars, where $\alpha = 1, \beta = 0$ corresponds to the 'spin-up' and $\alpha = 0, \beta = 1$ to the 'spin-down' eigenstates of conventional operator-based formulations. These plane wave spinors are superposed numerically to form a Gaussian packet with standard deviation $1 \times 10^{-24}\mathrm{kg\,m\,s^{-1}}$ in momentum space, giving a half-width of $1 \times 10^{-10}\mathrm{m}$ in real space. We take the particle's mass to be $1 \times 10^{-27}\mathrm{kg}$. After the impulse, the future evolution of ψ is found from equation (3.10). From this information we can analyse the behaviour of the relative spin-vector $s \wedge \gamma_0$, the probability density $J_0 = J \cdot \gamma_0$, and the streamlines for initial values of Φ.

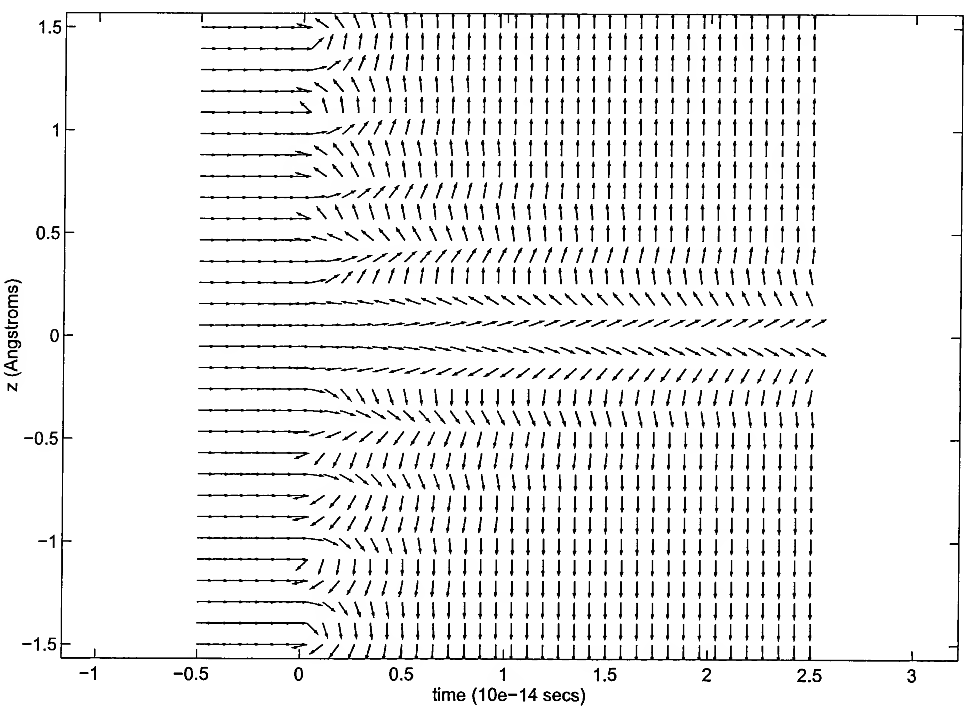
In Figure 1 we plot the evolution of J_0, of $s \wedge \gamma_0$, and of streamlines starting from various positions in the initial packet whose spin-vector initially points along the σ_1 direction ($\Phi = 1 - i\sigma_2$). The impulse imparts $\Delta p = 1 \times 10^{-23}\mathrm{kg\,m\,s^{-1}}$. The packet splits symmetrically into two equal sized packets and the streamlines bifurcate at the origin. The plot of the spin vector shows that immediately after the impulse, the spins are disordered because of the infinite rate of spin precession in the impulsive field. However, they sort themselves out in around $10^{-14}s$ such that, in the upward moving packet, the spin vector points along the field direction, whilst in the downward moving packet they are aligned in the opposite sense. Thus we recover the disjoint outcomes conventionally obtained by projection of the initial ψ onto eigenstates of the spin operators. As Dewdney *et al.* found, the choice of which packet a streamline enters is determined by its starting position in the initial packet. If, for example, the streamline is initially above the bifurcation point, it enters the upward-moving packet and the spin evolves along this path such that, on arrival at a particle detector, the spin points along the field direction. We see that, for this choice of Φ, each outcome is equally likely, as is shown by the symmetrical splitting of the packet.



(1a)

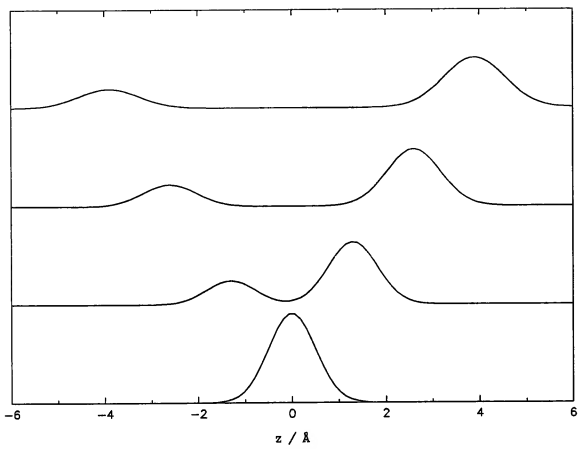


(1b)

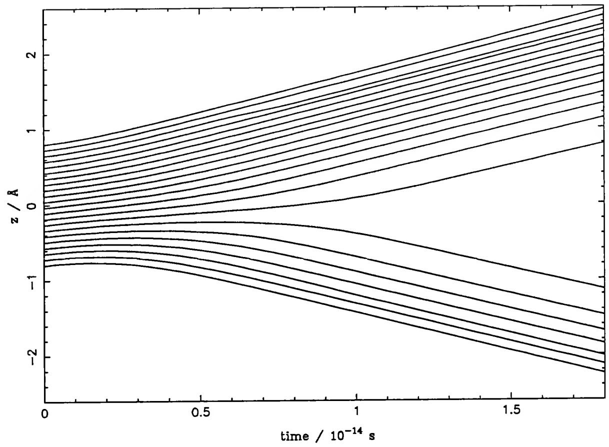


(1c)

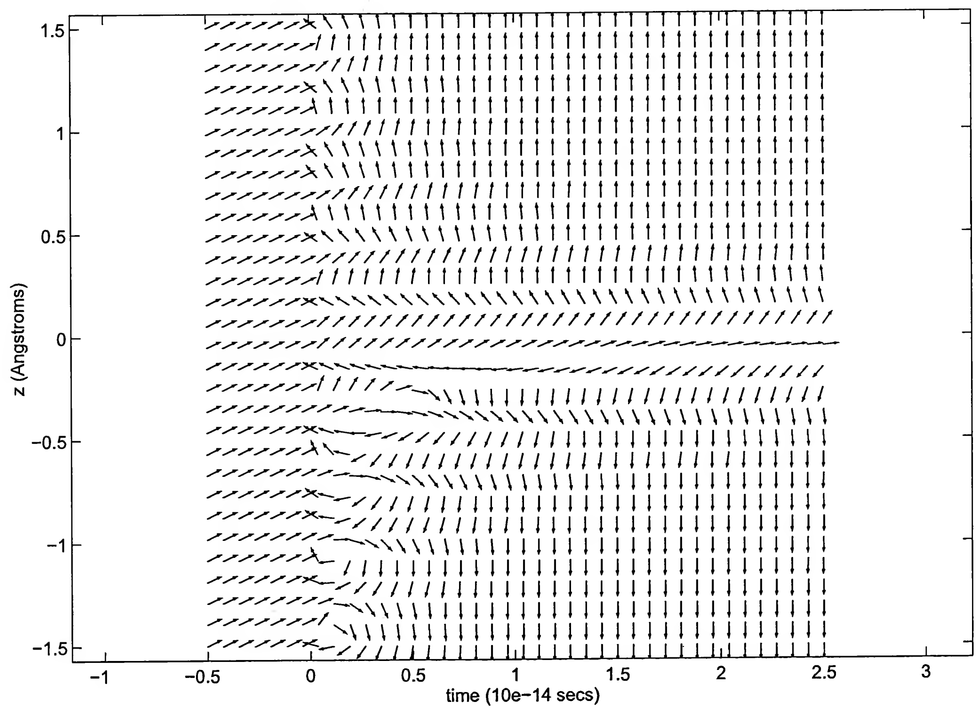
Figure 1: *Splitting of a wavepacket with* $\Phi = 1 - i\sigma_2$. *Figure 1a show the probability density,* J_0, *at* $t = 0, 1.3, 2.6, 3.9 \times 10^{-14}s$ *with* t *increasing up the figure, and Figure 1b the streamlines in the* (t, z) *plane. The evolution of the spin vector projected in the* (x, z) *plane is shown in Figure 1c.*



(2a)



(2b)



(2c)

Figure 2: *The asymmetric splitting of a wavepacket with* $\Phi = 1.618 - i\sigma_2$. *The bifurcation point is now below* $z = 0$ *so that more streamlines enter the upward moving packet. Figure 2c shows the evolution of* $s \wedge \gamma_0$ *projected in the* (x, z) *plane.*

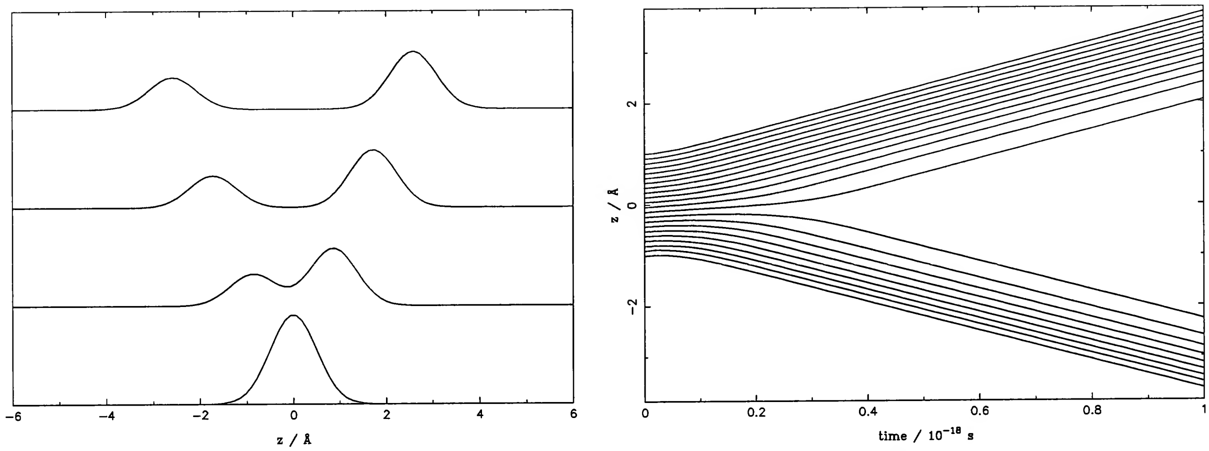
A calculation of the relative amplitudes of $\psi^{\uparrow}_{\pm}$ and $\psi^{\downarrow}_{\pm}$ shows that the negative energy packets are negligible for our chosen value of Δp, which is consistent with the observation that $\Delta p \ll m$ implying negligible pair production.

In Figure 2 we plot the evolution for unequal mixtures of 'spin-up' and 'spin-down' in the initial spinor. The impulse is as before. We now see an asymmetry in the splitting of the wavepacket, reflected in the lowering of the bifurcation point. More streamlines enter the upward moving packet and the density J_0 is correspondingly greater in this packet. We can calculate the probability for each of the two disjoint outcomes by numerically integrating J_0 over the spatial extent of each packet, giving a result in exact agreement with the results of the usual 'projection onto eigenstates' approach, as expected from our previous proof.

As a final illustration of our approach, we consider a large field gradient such that Δp is comparable to the mass of the particle. We suppose that the initial packet already has its spin aligned along the field gradient ($\Phi = 1$). For small field gradients we find that there is no splitting of the packet, all of the streamlines entering an upward-moving packet. However, for sufficiently large field gradients we find that a second packet is created which is composed of negative-energy states. In Figure 3 we plot the effects of an impulse $\Delta p = 1 \times 10^{-18}\mathrm{kg\,m\,s^{-1}}$. Both packets have spin aligned with the field gradient, but the packet composed of negative-energy states is deflected down the field gradient. The negative-energy states therefore behave as if they have a ratio of magnetic moment to mass opposite in sign to their positive-energy counterparts. In a field-theoretic treatment the appearance of negative-energy states would be attributed to pair production, with the antiparticles having the opposite sign of magnetic moment but positive mass and giving the opposite sign for their ratio, as we have observed.

5 Conclusions

Our aim in this letter has been to present a fully relativistic, causal account of the operation of a Stern-Gerlach apparatus. This analysis extends the calculations of Dewdney *et al.* [1] to the relativistic domain, but with the important difference that our model does not use any specific physical interpretation of quantum mechanics. Our plots of the streamlines and spin vectors are in qualitative agreement with those of Dewdney *et al.* in non-relativistic situations, despite their use of a current that is inconsistent with the Dirac current in such cases. However, our interpretation of the streamlines is very different from that of Bohm/de Broglie theory, where streamlines are taken to be the spacetime trajectories of a statistical ensemble of particles. Our use of streamlines is solely as a visual aid to monitor the flow of the density J^0. In the example considered here, the streamlines allow us to follow the flow of density from the initial wavepacket, through the region of interaction with the magnetic field, and into the final wavepackets that emerge from the apparatus. A similar use



(3a) (3b)

Figure 3: *Creation of negative-energy states by a large field gradient, giving an impulse of* $\Delta p = 1 \times 10^{-18} kgms^{-1}$. *The initial packet is entirely spin-up* ($\Phi = 1$) *and the resulting spin vector field (not shown) points uniformly along the* σ_3 *direction. Figure 3a is the probability density,* J_0, *plotted at* $t = 0, 3, 6, 9 \times 10^{-19} s$, *and Figure 3b shows the streamlines in the* (t, z) *plane. The small packet moving along the* $-\sigma_3$ *direction is composed of negative-energy states.*

of streamlines in tunnelling simulations have allowed us to make novel predictions for the duration of electron and photon tunnelling processes. These results will be presented in a forthcoming paper.

Our simulations make it clear that the Stern-Gerlach apparatus acts primarily as a polariser. It is the fact the dependence of the relative densities of the final state wavepackets on the incident spin that allows the experimenter to infer information about the incident spin. We could extend our simulations to include successive spin measurements, for example by directing particles in one output channel of an apparatus with the field along σ_3, into another apparatus with field along the σ_1 direction. This would not present any difficulty; we would simply find that the packet selected from the first apparatus would divide symmetrically at the second apparatus. Having developed a relativistic model for a single particle spin measurement, we intend to extend the treatment to describe correlated measurements made on pairs of particles with spacelike separation. This should shed much light on the issues of non-locality that EPR and other experiments raise.

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